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GALERKIN SINGLE-STEP METHODS FOR SECOND-ORDER HYPERBOLIC EQUATIONS

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Abstract—Single-step methods, coupled with Galerkin discretizations in space, are applied to second-order hyperbolic equations. These methods are applied directly to the second-order equations. Optimal order convergence estimates are derived

INTRODUCTION

We shall consider Galerkin, fully discrete approximations to the solution of the following initial boundary-value problem: Let Ω be a bounded domain in R^N with smooth $\partial\Omega$; we seek a real-valued function $u(x, t)$ satisfying

$$\begin{aligned} u_{tt} &= -Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(l_{ij}(x)\frac{\partial u}{\partial x_j}\right) - l_0(x)u \quad \text{in } \Omega \times (0, \tau], \\ u &= 0 \quad \text{on } \partial\Omega \times (0, \tau], \\ u(x, 0) &= u^0(x \quad \text{in } \Omega, \\ u_t(x, 0) &= u_t^0(x) \quad \text{in } \Omega; \end{aligned} \tag{1.1}$$

u^0, u_t^0 given. Here, $l_{ij}(x) = l_{ji}(x) \in C^\infty(\overline{\Omega})$, and $l_0 \in C^\infty(\overline{\Omega})$ with $l_0(x) \geq 0$ in $\overline{\Omega}$. We shall assume that the operator L is uniformly elliptic, i.e. for some constant $c_0 > 0$

$$\sum_{i,j=1}^{N} l_{ij}(x)\zeta_i\zeta_j \geq c_0 \sum_{i=1}^{N} \zeta_i^2, \quad \forall x \in \overline{\Omega}, \forall \zeta \in R^N.$$

In [1], Baker and Bramble analyzed both semidiscrete and fully discrete Galerkin approximations to the solution of (1.1). The fully discrete methods they considered are based on rational approximations to e^{-z}, and thus require converting the semidiscrete equations into a first-order system.

Another natural approach consists in applying special single-step methods directly to the second-order equation. Among such methods are the Runge–Kutta–Nyström methods (cf. [2]). This approach is more general, and it is well known that in the absence of the first time derivative, an extra order of accuracy can be gained for the same amount of work (see also [3], [4]). For problem (1.1), Gekeler[5] has obtained quasioptimal error estimates under regularity conditions on the initial data stronger than those in [1].

In this paper, we prove optimal order convergence estimates in L^2, for a general class of Nyström methods. In addition, the regularity requirements on the initial data are identical to those of [1]. Both conditionally and unconditionally stable methods are considered.

The paper is organized as follows. In Sec. 2, notation is established. In Sec. 3, the main stability result is proved. It is then shown that the optimal bound on the error can be obtained via an adaptation of the technique used in [1]. In Sec. 4, examples of fully discrete Nyström methods are discussed.

2. NOTATION AND PRELIMINARIES

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of L, and let $\{\phi_j\}_{j\geq 1}$ be the (orthonormal) set of corresponding eigenfunctions. Using these, we can define (cf. [1]) the following spaces: for $s \geq 0$,

$$\dot{H}^s(\Omega) = \left\{v: \|v\|_s \equiv \left\{\sum_{j=1}^{\infty} \lambda_j^s(v, \phi_j)^2\right\}^{1/2} < \infty\right\}.$$

It is shown in [6] that $\dot{H}^s(\Omega) = \{v \in H^s(\Omega), L^j v = 0 \text{ on } \partial\Omega, j < s/2\}$, where $H^s(\Omega)$ are the usual Sobolev spaces.

It is well known that for $s \geq 1$, if $u^0 \in \dot{H}^s(\Omega)$ and $u_t^0 \in \dot{H}^{s-1}(\Omega)$, then a unique solution to (1.1) exists for all $t > 0$ and that the following estimate holds:

$$\|u(t)\|_s^2 + \|u_t(t)\|_{s-1}^2 = \|u^0\|_s^2 + \|u_t^0\|_{s-1}^2. \tag{2.1}$$

We shall henceforth assume that the solution u is sufficiently smooth to guarantee the convergence estimates below.

Let $T\colon L^2 \to L^2$ be the solution operator of the associated elliptic problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

For integer $r \geq 2$, let $\{S_h^r\}_{h>0}$ be a one-parameter family of finite-dimensional subspaces of $L^2(\Omega)$, e.g. the space of piecewise polynomial functions defined on a triangulation of Ω. We assume the existence of a family $\{T_h\}_{h>0}$ of operators $T_h\colon L^2(\Omega) \to S_h^r$ possessing the following properties.

(i) T_h is symmetric, positive semidefinite on $L^2(\Omega)$ and positive definite on S_h^r.
(ii) $\|(T - T_h)f\| \leq ch^j \|f\|_{j-2}$, $2 \leq j \leq r$, where c is independent of h.
(iii) T_h has eigenvalues $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{d(h)}$. Moreover, $\mu_{d(h)} \leq c$ for some constant c independent of h.

Several types of approximating operators T_h satisfying (i)–(iii) are well known. These include the Galerkin method, two methods of Nitsche and the Lagrange multiplier method of Babuška (see [1,7–10]).

With T_h at hand, the semidiscrete approximation $u_h(t) \in S_h^r$ of $u(t)$ is defined by

$$\begin{aligned} T_h u_{htt}(t) + u_h(t) &= 0, \quad t > 0, \\ u_h(0) &= Pu^0, \\ u_{ht}(0) &= Pu_t^0, \end{aligned} \tag{2.2}$$

where $P\colon L^2 \to S_h^r$ is the L^2 projection operator.

Denoting the inverse of T_h on S_h^r by L_h, (2.2) can be rewritten as

$$\begin{aligned} u_{htt}(t) + L_h u_h(t) &= 0, \\ u_h(0) &= Pu^0, \\ u_{ht}(0) &= Pu_t^0. \end{aligned} \tag{2.3}$$

It is proven in [1,11] that for some constant c independent of h

$$\|u_h(t) - u(t)\| \leq ch^r(\|u^0\|_{r+1} + \|u_t^0\|_r). \tag{2.4}$$

Now let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the Euclidean inner product and norm on R^2. For $\mu > 0$, we define the following inner product on R^2

$$\langle x, y \rangle_\mu = x_1 y_1 + \mu^{-2} x_2 y_2$$

and let $|\cdot|_\mu$ denote the corresponding norm.

We shall also use the following seminorm on $L^2 \times L^2$: for $u = [u_1, u_2] \in L^2 \times L^2$, $|||u|||^2 = \|u_1\|^2 + \|T_h^{1/2} u_2\|^2$; note that $|||\cdot|||$ is a norm on $S_h^r \times S_h^r$.

Remark. All norm notations will also be used to denote corresponding operator norms.

3. FULLY DISCRETE APPROXIMATIONS

We consider operators $\mathcal{R}:R^2 \to R^2$ of the form

$$\mathcal{R}\,(x, z) = \begin{bmatrix} r_{11}(z) & x\, r_{12}(z) \\ -x^{-1}\, z\, r_{21}(z) & r_{22}(z) \end{bmatrix},$$

where x is a positive parameter and r_{ij} are rational functions in z with $r_{ij}(0) = 1$, $i, j = 1, 2$. These operators are used to generate fully discrete approximations to u, the solution of (1.1) in the following way: Let $k > 0$ be the time step, w_h^n, w_{ht}^n approximations in S_h^r to $u(nk)$ and $u_t(nk)$ respectively, then

$$\begin{aligned}\begin{bmatrix} w_h^{n+1} \\ w_{ht}^{n+1} \end{bmatrix} &= \mathcal{R}\,(k, k^2L_h) \begin{bmatrix} w_h^n \\ w_{ht}^n \end{bmatrix} \\ &= \begin{bmatrix} r_{11}(k^2L_h) & k\, r_{12}(k^2L_h) \\ -kL_h\, r_{21}(k^2L_h) & r_{22}(k^2L_h) \end{bmatrix} \begin{bmatrix} w_h^n \\ w_{ht}^n \end{bmatrix},\end{aligned} \tag{3.1}$$

with $w_h^0 = Pu^0$, $w_{ht}^0 = Pu_t^0$.

Formulation (3.1) is in general appropriate only for theoretical purposes. The practical implementation of these methods must take a computationally more efficient formulation. From this point of view, Runge–Kutta–Nyström methods form a particularly interesting class,

$$\begin{aligned} w_h^{n,i} &= w_h^n + k\alpha_i w_{ht}^n - k^2 \sum_{j=1}^{q} a_{ij} L_h w_h^{n,j}, \quad i = 1, \ldots, q \\ w_h^{n+1} &= w_h^n + k w_{ht}^n - k^2 \sum_{i=1}^{q} b_i L_h w_h^{n,i} \\ w_{ht}^{n+1} &= w_{ht}^n - k \sum_{i=1}^{q} \bar{b}_i L_h w_h^{n,i}\,; \end{aligned} \tag{3.2}$$

here q, α_i, a_{ij}, b_i, $\bar{b}_i$ are given constants and completely determine the method.

We next characterize the accuracy and stability properties of these methods. From (2.3) and (3.1), for $n \ge 0$,

$$\begin{aligned}\begin{bmatrix} \xi^{n+1} \\ \xi_t^{n+1} \end{bmatrix} &= \left[e^{-k\begin{bmatrix} 0 & -I \\ L_h & 0 \end{bmatrix}} - \mathcal{R}(k, k^2L_h) \right] \begin{bmatrix} \xi^n \\ \xi_t^n \end{bmatrix} \\ &= \begin{bmatrix} \cos kL_h^{1/2} - r_{11}(k^2L_h) & L_h^{-1/2}[\sin kL_h^{1/2} - kL_h^{1/2} r_{12}(k^2L_h)] \\ -L_h^{1/2}[\sin kL_h^{1/2} - kL_h^{1/2} r_{21}(k^2L_h)] & \cos kL_h^{1/2} - r_{22}(k^2L_h) \end{bmatrix} \begin{bmatrix} \xi^n \\ \xi_t^n \end{bmatrix},\end{aligned}$$

where $[\xi^n, \xi_t^n]^T = [u_h(t_n) - w_h^n, u_{ht}(t^n) - w_{ht}^n]^T$, $n \ge 0$. This motivates us to require $\mathcal{R}$ to satisfy the following consistency condition: there exist positive constants c, σ such that

$$\sum_{i=1}^{2} \{|\cos z - r_{ii}(z^2)| + |\sin z - z\, r_{i,3-i}(z^2)|\} \le cz^{\nu+1}, \quad 0 \le z \le \sigma \tag{3.3}$$

for some integer $\nu \ge 1$. We say that $\mathcal{R}$ is consistent of order ν.

We gather the stability conditions into four groups with $\sigma(\mathcal{R})$ denoting the spectral radius of $\mathcal{R}$.

Type I	(1) $\sigma(\mathcal{R}(1, z)) < 1$ for some $s < \infty$	$0 < z \le s$
	(2) $r_{11} = r_{22}$	
Type II	(1) $\sigma(\mathcal{R}(1, z)) \le 1$	$0 \le z \le s$
	(2) $r_{11} = r_{22}$	
	(3) $0 < \underline{b} \le \dfrac{\lvert r_{12}(z)\rvert}{\lvert r_{21}(z)\rvert} \le \bar{b} < \infty$	$0 < z \le s$

Type III (1) $\sigma(\mathscr{R}(1, z)) < 1,\ 0 < z \le s,$ for any finite s
(2) $r_{11} = r_{22}$
(3) $0 < \lim_{z\to\infty} \left|\frac{r_{12}(z)}{r_{21}(z)}\right| < \infty$

Type IV (1) $\sigma(\mathscr{R}(1, z)) \le 1$ $\quad 0 \le z < \infty$
(2) $r_{11} = r_{22}$
(3) $0 < \underline{b} \le \left|\frac{r_{12}(z)}{r_{21}(z)}\right| \le \bar{b} < \infty$ $\quad 0 < z < \infty.$

We next prove our main stability results.

THEOREM 3.1

Let $\mathscr{R}$ be of types I or II and suppose that $k^2\sigma(L_h) \le s$. Then there exists a constant c independent of h and k such that

$$|||\mathscr{R}^n(k, k^2L_h)||| \le c, \quad \forall n \ge 0.$$

Proof. Let $\{\phi_j\}_{j=1}^d \subset S_h^r$ be an orthonormal set of eigenfunctions of L_h, and $\{\lambda_j\}_{j=1}^d$ the set of corresponding eigenvalues arranged in nondecreasing order. Let $v_h^i, v_{ht}^i, i = 0, 1, \ldots, n$ in S_h^r be such that

$$\begin{bmatrix} v_h^n \\ v_{ht}^n \end{bmatrix} = (\mathscr{R}(k, k^2L_h))^n \begin{bmatrix} v_h^0 \\ v_{ht}^0 \end{bmatrix} \equiv \mathscr{R}^n(k, k^2L_h) \begin{bmatrix} v_h^0 \\ v_{ht}^0 \end{bmatrix}. \tag{3.4}$$

Letting $v_h^i = \Sigma_{j=1}^d \alpha_j^i \phi_j$, $v_{ht}^i = \Sigma_{j=1}^d \beta_j^i \phi_j$ by the orthonormality of $\{\phi_j\}_{j=1}^d$, we have

$$\begin{aligned}\begin{bmatrix} \alpha_j^n \\ \beta_j^n \end{bmatrix} &= \mathscr{R}^n(k, k^2\lambda_j) \begin{bmatrix} \alpha_j^0 \\ \beta_j^0 \end{bmatrix} \\ &= \begin{bmatrix} r_{11}(k^2\lambda_j) & kr_{12}(k^2\lambda_j) \\ -k\lambda_j r_{21}(k^2\lambda_j) & r_{22}(k^2\lambda_j) \end{bmatrix}^n \begin{bmatrix} \alpha_j^0 \\ \beta_j^0 \end{bmatrix}, \quad j = 1, 2, \ldots, d. \end{aligned} \tag{3.5}$$

Hence

$$\begin{bmatrix} \alpha_j^n \\ \lambda_j^{-1/2}\beta_j^n \end{bmatrix} = \tilde{\mathscr{R}}^n(k, k^2\lambda_j) \begin{bmatrix} \alpha_j^0 \\ \lambda_j^{-1/2}\beta_j^0 \end{bmatrix}, \tag{3.6}$$

where

$$\tilde{\mathscr{R}}(k, k^2\lambda_j) = \begin{bmatrix} r_{11}(k^2\lambda_j) & k\lambda_j^{1/2} r_{12}(k^2\lambda_j) \\ -k\lambda_j^{1/2} r_{21}(k^2\lambda_j) & r_{22}(k^2\lambda_j) \end{bmatrix}, \quad j = 1, 2, \ldots, d.$$

We first consider a type II method. For each j, $j = 1, \ldots, d$, let b_j be a nonzero scalar, to be suitably chosen below. Let $B_j = \operatorname{diag}\{1, b_j\}$. Consider $N_j = B_j \tilde{\mathscr{R}}(k, k^2\lambda_j)B_j^{-1}$. A simple computation shows that N_j is normal provided

$$|b_j|^2 = \left|\frac{r_{12}(k^2\lambda_j)}{r_{21}(k^2\lambda_j)}\right|. \tag{3.7}$$

Hence

$$\tilde{\mathscr{R}}(k, k^2\lambda_j) = B_j^{-1}U_j^*M_jU_jB_j,$$

where U_j is unitary and $M_j = \operatorname{diag}\{\mu_{1j}, \mu_{2j}\}$; μ_{1j}, μ_{2j} being the eigenvalues of $\tilde{\mathscr{R}}(k, k^2\lambda_j)$. These are also the eigenvalues of $\mathscr{R}(1, k^2\lambda_j)$, and hence $\max\{|\mu_{1j}|, |\mu_{2j}|\} \le 1$, $j =$

$1, 2, \ldots, d$. Thus

$$\begin{aligned} |\tilde{\mathscr{R}}^n(k, k^2\lambda_j)| &\le |B_j^{-1}|\,|B_j| \\ &\le \max\{1, |b_j|, |b_j|^{-1}\} \\ &\le \max\{\bar{b}, \underline{b}^{-1}\}. \end{aligned} \tag{3.8}$$

Next, suppose the method is of type I and fix $\delta > 0$ sufficiently small, so that in view of $r_{12}(0) = r_{21}(0) = 1$, we have for $\underline{b}, \bar{b} > 0$,

$$0 < \underline{b} \le \left|\frac{r_{12}(z)}{r_{21}(z)}\right| \le \bar{b} < \infty, \quad 0 \le z \le \delta.$$

If $k^2\lambda_j \le \delta$, then the argument leading to (3.8) can be used. If $k^2\lambda j > \delta$, then we let

$$\tilde{\mathscr{R}}(k, k^2\lambda_j) = U_j^* \begin{bmatrix} \mu_{1j} & c_j \\ 0 & \mu_{2j} \end{bmatrix} U_j,$$

where U_j is unitary. Since $\delta < k^2\lambda_j \le s$, and the method is of type I, $\max\{|\mu_{1j}|, |\mu_{2j}|\} \le \mu < 1$; it also holds that $|c_j| \le c$ independently of h and k. Hence

$$|\tilde{\mathscr{R}}^n(k, k^2\lambda_j)| = \left\|\begin{bmatrix} \mu_{1j}^n & c_j \sum_{l=0}^{n-1} \mu_{1j}^l \mu_{2j}^{n-1-l} \\ 0 & \mu_{2j}^n \end{bmatrix}\right\| \le c, \quad n \ge 1, \tag{3.9}$$

where c depends on δ and c_j but is otherwise independent of h and k.

To conclude the proof, we note that

$$\left|\left|\left|\begin{bmatrix} V_h^n \\ V_{ht}^n \end{bmatrix}\right|\right|\right|^2 = \sum_{j=1}^{d} ((\alpha_j^n)^2 + \lambda_j^{-1}(\beta_j^n)^2).$$

It follows then from (3.8) or (3.9) that

$$\begin{aligned} \left|\left|\left|\begin{bmatrix} V_h^n \\ V_{ht}^n \end{bmatrix}\right|\right|\right|^2 &\le c \sum_{j=1}^{d} ((\alpha_j^0)^2 + \lambda_j^{-1}(\beta_j^0)^2) \\ &= c \left|\left|\left|\begin{bmatrix} V_h^0 \\ V_{ht}^0 \end{bmatrix}\right|\right|\right|^2. \end{aligned}$$

THEOREM 3.2

Let $\mathscr{R}$ be of type III or IV. Then there exists a constant c independent of h and k such that

$$|||\mathscr{R}^n(k, k^2L_h)||| \le c.$$

Proof. The proof in the case of type IV methods can be done in a manner identical to the proof of a type II method, upon taking $s = \infty$.

For type III methods, the proof proceeds as follows. We write $[0, \infty) = [0, \beta) \cup [\beta, \infty)$, where β is chosen sufficiently large so that $r_{12}(z)/r_{21}(z)$ has neither zeros nor poles on $[\beta, \infty)$. In view of $0 < \lim_{z\to\infty}|r_{12}(z)/r_{21}(z)| < \infty$, we have $0 < \underline{b} \le |r_{12}(z)/r_{21}(z)| \le \bar{b} < \infty, \beta \le z < \infty$. Now on $[0, \beta)$, the proof proceeds exactly as in the case of type I methods, and on $[\beta, \infty)$ as in the case of type II methods. This completes the proof.

Remarks. (1) For type I and type II methods, a condition relating the sizes of h and k must be imposed. For subspaces S_h^r with elements satisfying inverse assumptions of the form

$$\|x\|_1 \leq ch^{-1}\|x\|,$$

it is easily shown that the aforementioned condition takes the form $kh^{-1} \leq c$ for some fixed c.

(2) Type II and type IV methods contain as special cases the methods considered in [1]. For these methods it can be shown that $r_{11} = r_{22}$ and $r_{12} = r_{21}$; it then follows that $\tilde{\mathscr{R}}^T(k, k^2\lambda_j)\tilde{\mathscr{R}}(k, k^2\lambda_j) = (r_{11}^2 + k^2\lambda_j r_{12}^2)I_{2\times 2}$. Now noting that $r_{11}^2 + k^2\lambda_j r_{12}^2 = (\sigma(\tilde{\mathscr{R}}(k, k^2\lambda_j)))^2 \leq 1$ for $k^2\lambda_j \leq s$, we can replace the constant c in Theorems 3.1 and 3.2 by 1, thus recovering the stability results of [1].

To derive the error estimates, we adopt the technique of [1]. For $t > 0$, $n \geq 1$, let

$$F_n(t, t^2y^2) = e^{-nt\begin{bmatrix} 0 & -1 \\ y^2 & 0 \end{bmatrix}} - \mathscr{R}^n(t, t^2y^2). \tag{3.10}$$

In the sequel, we shall take $s = \infty$ in the case of methods of type III or IV.

PROPOSITION 3.3

Let $\mathscr{R}$ have order ν and be of one of the types I–IV and suppose $k^2\sigma(L_h) \leq s$. Then there exists a constant c^*, independent of h and k, such that

$$|F_n(t, t^2y^2)|_y \leq nc^*(ty)^l, \quad n \geq 1, \tag{3.11}$$

for $ty \leq s$ and $1 \leq l \leq \nu + 1$.

Proof. Let $\sigma^* = \min(\sigma, 1)$. We have after a simple calculation

$$S^{-1}F_1(t, t^2y^2)S = \begin{bmatrix} \cos ty - r_{11}(t^2y^2) & \sin ty - tyr_{12}(t^2y^2) \\ -\sin ty + tyr_{21}(t^2y^2) & \cos ty - r_{22}(t^2y^2) \end{bmatrix}, \tag{3.12}$$

where $S = \mathrm{diag}\{1, y\}$. It follows from (3.3) that

$$\begin{aligned} |F_1(t, t^2y^2)|_y &= |S^{-1}F_1(t, t^2y^2)S| \\ &\leq c(ty)^{\nu+1} \leq c(ty)^l, \quad ty \leq \sigma^*, \ 1 \leq l \leq \nu + 1. \end{aligned} \tag{3.13}$$

Now suppose $\sigma^* < s$. Noting that,

$$\left|e^{-jt\begin{bmatrix} 0 & -1 \\ y^2 & 0 \end{bmatrix}}\right|_y = 1, \quad j \geq 0, \ t, y > 0; \tag{3.14}$$

for $\sigma^* < ty \leq s$, we have, using Theorem 3.1 or Theorem 3.2,

$$\begin{aligned} |F_1(t, t^2y^2)|_y \leq c &= c(ty)^l \left(\frac{\sigma^*}{ty}\right)^l (\sigma^*)^{-l} \\ &\leq c(ty)^l(\sigma^*)^{-(\nu+1)} \leq c(ty)^l. \end{aligned} \tag{3.15}$$

Now for any integer $n \geq 1$, from Theorem 3.1 or Theorem 3.2, (3.13), (3.14) and (3.15), for $ty \leq s$ we have

$$\begin{aligned} |F_n(t, t^2y^2)|_y &= \left|\sum_{j=0}^{n-1} \mathscr{R}^j(t, t^2y^2)F_1(t, t^2y^2)e^{-(n-1-j)t\begin{bmatrix} 0 & -1 \\ y^2 & 0 \end{bmatrix}}\right|_y \\ &\leq cn|F_1(t, t^2y^2)|_y \leq nc^*(ty)^l, \quad 1 \leq l \leq \nu + 1. \end{aligned}$$

PROPOSITION 3.4

Under the conditions of Proposition 3.3, there exists a constant c independent of h and k such that for all $z \in L^2 \times L^2$

$$|||F_n(k, k^2L_h) \begin{bmatrix} 0 & -T_h \\ I & 0 \end{bmatrix}^l z||| \leq ck^{l-1}|||z|||, \quad 1 \leq l \leq \nu + 1 \tag{3.16}$$

Proof. We first prove (3.16) for l even. Letting

$$z = \sum_{j=1}^{d} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} \phi_j \equiv \sum_{j=1}^{d} \Gamma_j \phi_j,$$

we find

$$E_n \equiv F_n(k, k^2L_n) \begin{bmatrix} 0 & -T_h \\ I & 0 \end{bmatrix}^l z = (-1)^{l/2} F_n(k, k^2L_h) \begin{bmatrix} T_h^{l/2} & 0 \\ 0 & T_h^{l/2} \end{bmatrix} z$$
$$= (-1)^{l/2} \sum_{j=1}^{d} \lambda_j^{-l/2} F_n(k, k^2L_h)\Gamma_j\phi_j.$$

It is easily verified that

$$|||E_n|||^2 = \sum_{j=1}^{d} \lambda_j^{-l} |F_n(k, k^2\lambda_j)\Gamma_j|_{\lambda_j^{1/2}}^2$$
$$\leq \sum_{j=1}^{d} \lambda_j^{-l} |F_n(k, k^2\lambda_j)|_{\lambda_j^{1/2}}^2 |\Gamma_j|_{\lambda_j^{1/2}}^2. \tag{3.17}$$

Since

$$|||z|||^2 = \sum_{j=1}^{d} |\Gamma_j|_{\lambda_j^{1/2}}^2,$$

from (3.11) it follows that

$$|||E_n|||^2 \leq \sum_{j=1}^{d} \lambda_j^{-l} n^2 (c^*)^2 (k\lambda_j^{1/2})^{2l} |\Gamma_j|_{\lambda_j^{1/2}}^2$$
$$\leq (c^*)^2 k^2 n^2 k^{2l-2} \sum_{j=1}^{d} |\Gamma_j|_{\lambda_j^{1/2}}^2$$
$$\leq ck^{2l-2}|||z|||^2. \tag{3.18}$$

Now for l odd, we have

$$|||E_n|||^2 = \left|\left|\left| F_n(k, k^2L_h) \begin{bmatrix} 0 & -T_h \\ I & 0 \end{bmatrix}^{l+1} \begin{bmatrix} 0 & -T_h \\ I & 0 \end{bmatrix}^{-1} z \right|\right|\right|^2$$
$$= \left|\left|\left| F_n(k, k^2L_n) \begin{bmatrix} 0 & -T_h \\ I & 0 \end{bmatrix}^{l+1} z^1 \right|\right|\right|^2,$$

where

$$z^1 = \sum_{j=1}^{d} \begin{bmatrix} \alpha_j^1 \\ \beta_j^1 \end{bmatrix} \phi_j \equiv \sum_{j=1}^{d} \Gamma_j^1 \phi_j, \quad \text{with } \alpha_j^1 = \beta_j,\ \beta_j^1 = -\lambda_j\alpha_j.$$

Hence, from (3.17) and (3.11),

$$\begin{aligned}|||E_n|||^2 &\le \sum_{j=1}^{d} \lambda_j^{-(l+1)} |F_n(k, k^2\lambda_j)|^2_{\lambda_j^{1/2}} |\Gamma_j^1|^2_{\lambda_j^{1/2}} \\ &\le \sum_{j=1}^{d} \lambda_j^{-(l+1)} (nc^*)^2 (k\lambda_j^{1/2})^{2l} |\Gamma_j^1|^2_{g\lambda_j^{1/2}}.\end{aligned}$$

Now since

$$\sum_{j=1}^{d} \lambda_j^{-1} |\Gamma_j^1|^2_{\lambda_j^{1/2}} = |||z|||^2,$$

the result follows.

With Propositions 3.3 and 3.4 at hand, we can use Theorem 3.1 of [1]. So let $E_n^T = [u_h(nk) - w_h^n, u_{ht}(nk) - w_{ht}]^T$, where $[u_h(nk), u_{ht}(nk)]^T$ and $[w_h^n, w_{ht}^n]^T$ are given by (2.2) and (3.1), respectively. We have the following.

THEOREM 3.5

Under the conditions of Proposition 3.3, there exists a constant c independent of h and k such that

$$\sup_{0 \le n \le T/k} |||E_n||| \le ch^r[\|u^0\|_{r+1} + \|u_t^0\|_r] + ck^\nu[\|u^0\|_{\nu+1} + \|u_t^0\|_\nu]. \tag{3.19}$$

Remark. Using (3.19) with (2.4) we get

$$\begin{aligned}\sup_{0 \le n \le T/k} \|u(nk) - w_h^n\| \le\ & ch^r[\|u^0\|_{r+1} + \|u_t^0\|_r] \\ &+ ck^\nu[\|u^0\|_{\nu+1} + \|u_t^0\|_\nu].\end{aligned} \tag{3.20}$$

Moreover, similar convergence estimates can be obtained for any initial data $[w_h^0, w_{ht}^0]^T$ satisfying $\|u^0 - w_h^0\| + \|u_t^0 - w_{ht}^0\| \le ch^r$.

4. EXAMPLES

We now consider two examples of Nyström methods as given by (3.2). First, letting $(A)_{ij} = a_{ij}$, $b^T = (b_1, \ldots, b_q)$, $\bar{b}^T = (\bar{b}_1, \ldots, \bar{b}_q)$, $e^T = (1, 1, \ldots, 1)$, $\alpha^T = (\alpha_1, \ldots, \alpha_q)$, after simple calculations

$$\begin{aligned}&r_{11}(z) = 1 - zb^T(I + zA)^{-1}e,\ r_{22}(z) = 1 - z\bar{b}^T(I + zA)^{-1}\alpha \\ &r_{12}(z) = 1 - zb^T(I + zA)^{-1}\alpha,\ r_{21}(z) = \bar{b}^T(I + zA)^{-1}e.\end{aligned} \tag{4.1}$$

We first consider an explicit method (cf. [4,5]). Here $q = 3$, $\nu = 4$, $s = 6$:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1/8 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}, \quad \alpha^T = (0, 1/2, 1),$$

$$b^T = (\frac{1}{6}, \frac{1}{3}, 0), \quad \bar{b}^T = \left[\frac{1}{6}, \frac{4}{6}, \frac{1}{6}\right];$$

$$r_{11}(z) = r_{22}(z) = 1 - \frac{z}{2} + \frac{z^2}{24},$$

$$r_{12}(z) = 1 - \frac{z}{6},$$

$$r_{21}(z) = 1 - \frac{z}{6} + \frac{z^2}{96}.$$

The implementation of this method requires three matrix-vector multiplications (function evaluations) per step, and solutions of three linear systems $Gx = \rho$, where G is the Gramian matrix. The corresponding fourth-order polynomial approximation to e^{-z}, on the other hand, requires four function evaluations and solutions of four linear systems.

We next consider a two-stage fourth-order implicit method. Let $\beta = 1/6\,(1 + \sqrt{3/2})$ (largest root of $144\beta^2 - 48\beta + 1$). For $\gamma \neq \beta$, we have the one-parameter family of methods

$$A = \begin{bmatrix} \beta & 0 \\ \gamma - \beta & \beta \end{bmatrix}, \quad b^T = (\gamma - \beta)^{-1}\left(\frac{\gamma}{2} - \frac{1}{24}, -\frac{\beta}{2} + \frac{1}{24}\right),$$

$$\bar{b}^T = (\gamma - \beta)^{-1}\left(\gamma - \frac{1}{6}, -\beta + \frac{1}{6}\right), \quad \alpha^T = \left(2\beta + \frac{1}{6}, 2\gamma + \frac{1}{6}\right).$$

This method requires two function evaluations and solutions of two linear systems with the same coefficient matrix $(G + \beta k^2 S)$, where S is the stiffness matrix. However, this method is only conditionally stable (type II) for $s < 12$.

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